On inequalities for normalized Schur functions*

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Abstract

We prove a conjecture of Cuttler et al. [2011] [A. Cuttler, C. Greene, and M. Skandera; *Inequalities for symmetric means.* European J. Combinatorics, 32(2011), 745–761] on the monotonicity of *normalized Schur functions* under the usual (dominance) partial-order on partitions. We believe that our proof technique may be helpful in obtaining similar inequalities for other symmetric functions.

We prove a conjecture of Cuttler et al. [2011] on the monotonicity of normalized Schur functions under the majorization (dominance) partial-order on integer partitions.

Schur functions are one of the most important bases for the algebra of symmetric functions. Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be a tuple of n real variables. Schur functions of $\boldsymbol{x}$ are indexed by integer partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$, where $\lambda_1 \geq \cdots \geq \lambda_n$, and can be written as the following ratio of determinants [Schur, 1901, pg. 49], [Macdonald, 1995, (3.1)]:

$$s_\lambda(\boldsymbol{x}) = s_\lambda(x_1, \ldots, x_n) := \frac{\det([x_i^{\lambda_j+n-j}]_{i,j=1}^n)}{\det([x_i^{n-j}]_{i,j=1}^n)}. \tag{0.1}$$

To each Schur function $s_\lambda(\boldsymbol{x})$ we can associate the *normalized Schur function*

$$S_\lambda(\boldsymbol{x}) \equiv S_\lambda(x_1, \ldots, x_n) := \frac{s_\lambda(x_1, \ldots, x_n)}{s_\lambda(1, \ldots, 1)} = \frac{s_\lambda(\boldsymbol{x})}{s_\lambda(1^n)}. \tag{0.2}$$

Let λ, $\mu \in \mathbb{R}^n$ be decreasingly ordered. We say λ is *majorized* by μ, denoted $\lambda \prec \mu$, if

$$\sum\nolimits_{i=1}^k \lambda_i = \sum\nolimits_{i=1}^k \mu_i \quad \text{for } 1 \leq i \leq n-1, \quad \text{and} \quad \sum\nolimits_{i=1}^n \lambda_i = \sum\nolimits_{i=1}^n \mu_i. \tag{0.3}$$

Cuttler et al. [2011] studied normalized Schur functions (0.2) among other symmetric functions, and derived inequalities for them under the partial-order (0.3). They also conjectured related inequalities, of which perhaps Conjecture 1 is the most important.

Conjecture 1 ([Cuttler et al., 2011])**.** *Let λ and μ be partitions; and let $\boldsymbol{x} \geq 0$. Then,*

$$S_\lambda(\boldsymbol{x}) \leq S_\mu(\boldsymbol{x}), \qquad \textit{if and only if} \quad \lambda \prec \mu.$$

Cuttler et al. [2011] established necessity (i.e., $S_\lambda \leq S_\mu$ only if $\lambda \prec \mu$), but sufficiency was left open. We prove sufficiency in this paper.

Theorem 2. *Let λ and μ be partitions such that $\lambda \prec \mu$, and let $\boldsymbol{x} \geq 0$. Then,*

$$S_\lambda(\boldsymbol{x}) \leq S_\mu(\boldsymbol{x}).$$

Our proof technique differs completely from [Cuttler et al., 2011]: instead of taking a direct algebraic approach, we invoke a well-known integral from random matrix theory. We believe that our approach might extend to yield inequalities for other symmetric polynomials such as Jack polynomials [Jack, 1970] or even Hall-Littlewood and Macdonald polynomials [Macdonald, 1995].

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1 Majorization inequality for Schur polynomials

Our main idea is to represent normalized Schur polynomials (0.2) using an integral compatible with the partial-order '$\prec$'. One such integral is the *Harish-Chandra-Itzykson-Zuber (HCIZ)* integral [Harish-Chandra, 1957, Itzykson and Zuber, 1980]:

$$I(A,B) := \int_{U(n)} e^{\mathrm{tr}(U^*AUB)} dU = c_n \frac{\det([e^{a_i b_j}]_{i,j=1}^n)}{\Delta(\boldsymbol{a})\Delta(\boldsymbol{b})}, \tag{1.1}$$

where dU is the Haar probability measure on the unitary group $U(n)$; $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors of eigenvalues of the Hermitian matrices A and B; Δ is the *Vandermonde determinant* $\Delta(\boldsymbol{a}) := \prod_{1\le i<j\le n}(a_j - a_i)$; and c_n is the constant

$$c_n = \left(\prod_{i=1}^{n-1} i!\right) = \Delta([1,\dots,n]) = \prod_{1\le i<j\le n}(j-i). \tag{1.2}$$

The following observation [Harish-Chandra, 1957] is of central importance to us.

Proposition 3. *Let A be a Hermitian matrix, λ an integer partition, and B the diagonal matrix* $\mathrm{Diag}([\lambda_j + n - j]_{j=1}^n)$*. Then,*

$$\frac{s_\lambda(e^{a_1},\dots,e^{a_n})}{s_\lambda(1,\dots,1)} = \frac{1}{E(A)} I(A,B), \tag{1.3}$$

where the product $E(A)$ is given by

$$E(A) = \prod_{1\le i<j\le n} \frac{e^{a_i} - e^{a_j}}{a_i - a_j}. \tag{1.4}$$

Proof. Recall from Weyl's dimension formula that

$$s_\lambda(1,\dots,1) = \prod_{1\le i<j\le n} \frac{(\lambda_i - i) - (\lambda_j - j)}{j-i}. \tag{1.5}$$

Now use identity (1.5), definition (1.2), and the ratio (0.1) in (1.1), to obtain (1.3). □

Assume without loss of generality that for each i, $x_i > 0$ (for $x_i = 0$, apply the usual continuity argument). Then, there exist reals $a_1,\dots,a_n$ such that $e^{a_i} = x_i$, whereby

$$S_\lambda(x_1,\dots,x_n) = \frac{s_\lambda(e^{\log x_1},\dots,e^{\log x_n})}{s_\lambda(1,\dots,1)} = \frac{I(\log X, B(\lambda))}{E(\log X)}, \tag{1.6}$$

where $X = \mathrm{Diag}([x_i]_{i=1}^n)$; we write $B(\lambda)$ to explicitly indicate B's dependence on λ as in Prop. 3. Since $E(\log X) > 0$, to prove Theorem 2, it suffices to prove Theorem 4 instead.

Theorem 4. *Let X be an arbitrary Hermitian matrix. Define the map $F : \mathbb{R}^n \to \mathbb{R}$ by*

$$F(\lambda) := I(X, \mathrm{Diag}(\lambda)), \qquad \lambda \in \mathbb{R}^n.$$

Then, F is Schur-convex, i.e., if $\lambda, \mu \in \mathbb{R}^n$ such that $\lambda \prec \mu$, then $F(\lambda) \le F(\mu)$.

Proof. We know from [Marshall et al., 2011, Proposition C.2, pg. 97] that a convex and symmetric function is Schur-convex. From the HCIZ integral (1.1) symmetry of F is apparent; to establish its convexity it suffices to demonstrate midpoint convexity:

$$F\left(\tfrac{\lambda+\mu}{2}\right) \leq \tfrac{1}{2}F(\lambda) + \tfrac{1}{2}F(\mu) \qquad \text{for} \quad \lambda, \mu \in \mathbb{R}^n. \tag{1.7}$$

The elementary manipulations below show that inequality (1.7) holds.

$$\begin{aligned}
F\left(\tfrac{\lambda+\mu}{2}\right) &= \int_{U(n)} \exp\big(\mathrm{tr}\big[U^* X U \,\mathrm{Diag}\big(\tfrac{\lambda+\mu}{2}\big)\big]\big)\,dU \\
&= \int_{U(n)} \exp\big(\mathrm{tr}\big[\tfrac{1}{2}U^* X U \,\mathrm{Diag}(\lambda) + \tfrac{1}{2}U^* X U \,\mathrm{Diag}(\mu)\big]\big)\,dU \\
&= \int_{U(n)} \sqrt{\exp\big(\mathrm{tr}[U^* X U \,\mathrm{Diag}(\lambda)]\big) \cdot \exp\big(\mathrm{tr}[U^* X U \,\mathrm{Diag}(\mu)]\big)}\,dU \\
&\leq \int_{U(n)} \big(\tfrac{1}{2}\exp\big(\mathrm{tr}[U^* X U \,\mathrm{Diag}(\lambda)]\big) + \tfrac{1}{2}\exp\big(\mathrm{tr}[U^* X U \,\mathrm{Diag}(\mu)]\big)\big)\,dU \\
&= \tfrac{1}{2}F(\lambda) + \tfrac{1}{2}F(\mu),
\end{aligned}$$

where the inequality follows from the arithmetic-mean geometric-mean inequality. □

Corollary 5. *Conjecture 1 is true.*

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